Semicontinuity properties of Kazhdan-Lusztig cells in affine Weyl groups of rank 2

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- Robinson-Schensted correspondence ($\sim$ 1930's)
  - $\sharp$ elements in a cell =dimension of some irreducible in S_n
- Joseph's work on primitive ideals in $U(\mathfrak{g})$ ($\sim$ 1970/80's)
  - Each left cell is a basis of a representation of W (Weyl group)
- Kazhdan-Lusztig : Elementary definition for arbitrary Coxeter group W ($\sim$ 1979)
  - Give rise to representation of W and of the Hecke algebra
  - Geometric interpretation in term of intersection cohomology
- Lusztig : Cells in the multiparameter case (1983/2003)
  - Formulation of conjectures **P1–P15**
  - Conjectural interpretation in term of character sheaves on disconnected groups

Coxeter group (W, S): generating set S, relations

$$(ss')^{m_{s,s'}} = 1 \text{ where } m_{s,s} = 1 \text{ and } m_{s,s'} \geq 2.$$

Let V be a vector space with basis $\{e_s | s \in S\}$ and

$$B(e_s, e_{s'}) = -\cos(\frac{\pi}{m_{s,s'}})$$

To each s we associate the reflection $\sigma_s(x) = x - 2B(e_s, x)e_s$.

B is positive definite $\Leftrightarrow$ W is a finite reflection group

"tame": B is positive, "integral": $m_{s,s'} \in \{2, 3, 4, 6, \infty\}$

There are 3 kinds of tame Coxeter groups

- finite and integral (Weyl groups)
- finite and non integral
- Infinite and automatically integral (affine Weyl groups)

Let W be a Coxeter group with generating set S and ℓ be the usual length function. A weight function $L : W \to \mathbb{N}$ is a function such that

$$L(ww') = L(w) + L(w') \text{ whenever } \ell(ww') = \ell(w) + \ell(w')$$

- ℓ is a weight function ("equal parameter case")
- A weight function L is completely determined by its values on S
- $L(s) = L(t)$ if s and t are conjuate

Let $\mathcal{H} = \mathcal{H}_{\mathcal{A}}(W, S, L)$ be the corresponding Iwahori–Hecke algebra over $A = \mathbb{Z}[v, v^{-1}]$. Thus, $\mathcal{H}$ is $\mathcal{A}$-free with basis $\{T_w \mid w \in W\}$. For $s \in S$ and $w \in W$, the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + (v^{L(s)} - v^{-L(s)}) T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

There is a unique involution $h \mapsto \bar{h}$ of $\mathcal{H}$ such that

$$\bar{v} = v^{-1} \quad \text{and} \quad \overline{T_w} = T_{w^{-1}}^{-1}$$

Theorem. Kazhdan-Lusztig ($\sim$ 1979) Lusztig ($\sim$ 1983)

For any $w \in W$, there exists a unique $C_w \in \mathcal{H}$ such that:

- $\overline{C_w} = C_w$
- $C_w = \sum_{y \in W} P_{y,w} T_y$ where $P_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}]$

The C_w's form an $\mathcal{A}$-basis of $\mathcal{H}$.

- $C_1 = T_1$
- $C_s = T_s + v^{-L(s)} T_1$

Let $\leq_{\mathcal{L}}$ be the preorder generated by

$$x \leq_{\mathcal{L}} y \iff C_x \text{ appears in } hC_y \text{ for some } h$$

We write $x \sim_{\mathcal{L}} y$ if $x \leq_{\mathcal{L}} y$ and $y \leq_{\mathcal{L}} x$. This defines an equivalence relation, the equivalence classes are called *left cells*.

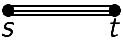
Similarly one can define

- $\leq_{\mathcal{R}}$ multiplying on the right (right cells)
- $\leq_{\mathcal{LR}}$ multiplying on both sides (two-sided cells)

One can show that $x \leq_{\mathcal{L}} y \Leftrightarrow x^{-1} \leq_{\mathcal{R}} y^{-1}$.

$$\mathcal{H}C_w \subseteq \sum_{y \leq_{\mathcal{L}} w} \mathcal{A}C_y, \qquad C_w\mathcal{H} \subseteq \sum_{y \leq_{\mathcal{R}} w} \mathcal{A}C_y, \qquad \mathcal{H}C_w\mathcal{H} \subseteq \sum_{y \leq_{\mathcal{LR}} w} \mathcal{A}C_y.$$

$$W = \coprod \{ \text{ left cells } \} = \coprod \{ \text{ right cells } \} = \coprod \{ \text{ two-sided cells } \}.$$

W be of type G_2 i.e 

$L(s) = a \in \mathbb{N}$ and $L(t) = b \in \mathbb{N}$

	left cells
$a > b$	$\{1\}$
	$\{t\}$
	$\{st, tst, stst, tstst\}$
	$\{s, ts, sts, tsts\}$
	$\{ststs\}$
	$\{ststst\}$
$a = b$	$\{1\}$
	$\{t, st, tst, stst, tstst\}$
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$C_s C_{tsts} = C_{ststs} + (v^{a-b} + v^{b-a}) C_{sts} + C_s$
and $tsts \leq_{\mathcal{L}} s$ thus $\{s, ts, sts, tsts\} \subset$ left cell.

$C_s C_{tstst} = C_{ststst} + (v^{a-b} + v^{b-a}) C_{stst} + C_{st}$
and $tstst \leq_{\mathcal{L}} st$ thus $\{st, tst, stst, tstst\} \subset$ left cell.

$C_s C_{tsts} = C_{ststs} + C_{sts}$
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Structure constants: Write

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z \text{ where } h_{x,y,z} \in \mathcal{A}.$$

G. Lusztig (1985): Define function $\mathbf{a}\colon W \to \mathbb{N}_0$ by

$$\mathbf{a}(z) = \min\{i \geq 0 \mid v^{-i} h_{x,y,z} \in \mathbb{Z}[v^{-1}]\ \forall x, y \in W\}.$$

Lusztig formulated 15 conjectures including

- If $x \sim_{\mathcal{LR}} y$ then $\mathbf{a}(x) = \mathbf{a}(y)$
- If $x \leq_{\mathcal{LR}} y$ and $\mathbf{a}(x) = \mathbf{a}(y)$ then $x \sim_{\mathcal{LR}} y$
- If $x \leq_{\mathcal{L}} y$ and $x \sim_{\mathcal{LR}} y$ then $x \sim_{\mathcal{L}} y$

  Note that this would imply that the two-sided cells are the smallest subsets which are at the same time union of left cells and union of right cells

Extremely hard to compute even in small groups!

Let (W, S) be a Coxeter group. Assume that $S = S_1 \cup S_2$ where no element of S_1 is conjugate to an element of S_2. Let

$$L_{a,b}(s) = a \text{ for all } s \in S_1 \text{ and } L_{a,b}(t) = b \text{ for all } t \in S_2$$

We denote by $\mathcal{L}_{a,b}(W)$ the partition of W into cells wrt $L_{a,b}$.

Semicontinuity Conjecture (1). BONNAFÉ ($\sim$ 2007)

There exist $r_1 < \ldots < r_n \in \mathbb{Q}$ such that

1. If a, b, a', b' are such that $r_i < a/b, a'/b' < r_{i+1}$ then $\mathcal{L}_{a,b}(W)$ and $\mathcal{L}_{a',b'}(W)$ coincide.
2. If $a/b = r_i$ then the partition into cells is given by taking the smallest common refinement of the partition $\mathcal{L}_{a,b}(W)$ (where $r_{i-1} < a/b < r_i$) and $\mathcal{L}_{a',b'}(W)$ (where $r_i < a'/b' < r_{i+1}$)

W be of type G_2 i.e 

$L(s) = a \in \mathbb{N}$ and $L(t) = b \in \mathbb{N}$

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(W, S) such that $S = S_1 \cup S_2$...

Semicontinuity Conjecture (2). Bonnafé ($\sim$ 2007)

The cells of $\mathcal{L}_{a,0}(W)$ are the smallest subset of W which are

- union of cells of $\mathcal{L}_{a,b}(W)$ where $a \gg b$
- stable by multiplication by W_{S_2}

Cells of $\mathcal{L}_{a,0}(W)$: Set $\widetilde{S_1} := \{wtw^{-1} | w \in W_{S_2}, t \in S_1\}$.

Let $\widetilde{W}$ be the subgroup of W generated by $\widetilde{S_1}$. Then $(\widetilde{W}, \widetilde{S_1})$ is a Coxeter group and we have

$$W = W_{S_2} \ltimes \tilde{W}$$

Let $\tilde{L}$ be the weight function on $\widetilde{S_1}$ such that $\tilde{L}(wtw^{-1}) = L_{a,0}(t)$.

The cells of W with respect to $L_{a,0}$ are of the form

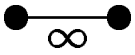
- $W_{S_2}.C$ where C is a left cell of $(\tilde{W}, \tilde{L})$
- $C.W_{S_2}$ where C is a right cell of $(\tilde{W}, \tilde{L})$
- $W_{S_2}.C.W_{S_2}$ where C is a two-sided cell of $(\tilde{W}, \tilde{L})$

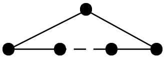
Theorem. G. ($\sim$ 2008)

Assume that $a \gg b$. Then the cells of W_{S_2} considered as a proper Coxeter group are cells in the group W.

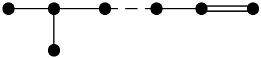
The semicontinuity conjecture holds for

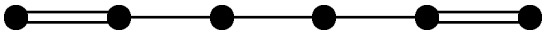
- dihedral groups (Geck-Pfeiffer, Lusztig);
- type F_4 (Geck);
- affine Weyl groups of rank 2.

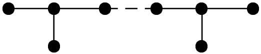
$\tilde{A}_1$:


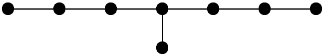
$\tilde{A}_n$:


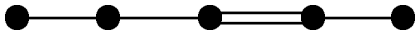
$\tilde{B}_2 = \tilde{C}_2$:

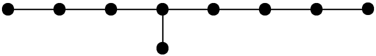

$\tilde{B}_n$:


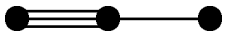
$\tilde{C}_n$:


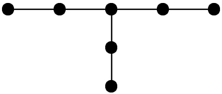
$\tilde{D}_n$:


$\tilde{E}_7$:


$\tilde{F}_4$:


$\tilde{E}_8$:


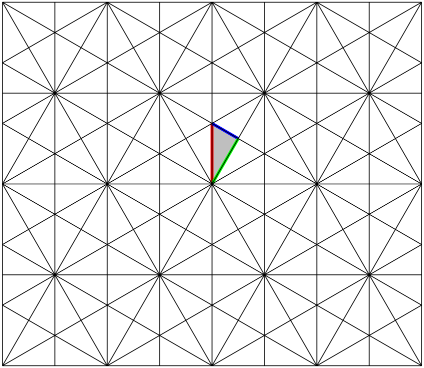
$\tilde{G}_2$:


$\tilde{E}_6$:


All proper parabolic subgroups are finite

Alcoves: Connected component of $V - \{\text{hyperplans}\}$. Fix an alcove A_0

A face is a co-dimension 1 facet of an alcove. Example : faces of A_0.



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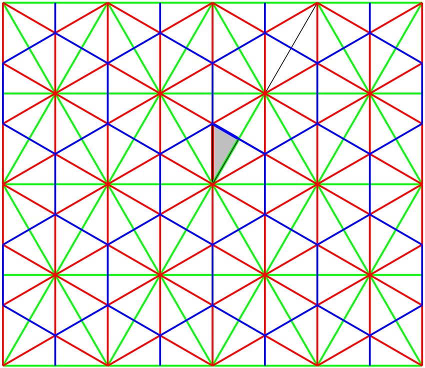

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Here we have 3 orbits, namely :

$$s_1 = \text{green}$$

$$s_2 = \text{red}$$

$$s_3 = \text{blue}$$

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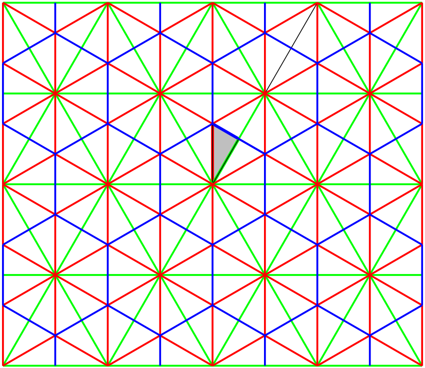

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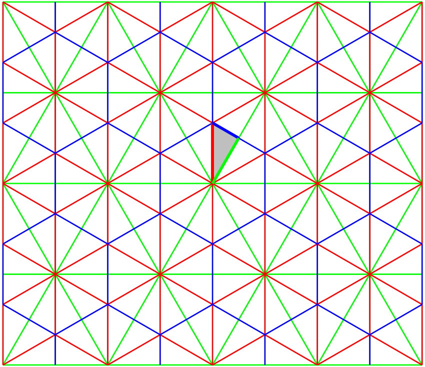
For $s \in S$, we define an involution $A \mapsto sA$ of X, where sA is the unique alcove which shares with A a face of type s. The set of such map is a group of permutation of X which is a Coxeter group W.

We identify $w \in W$ with the alcove wA_0.



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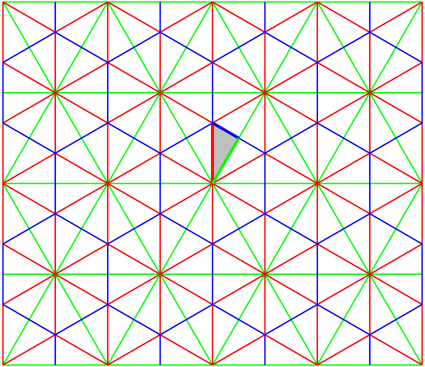

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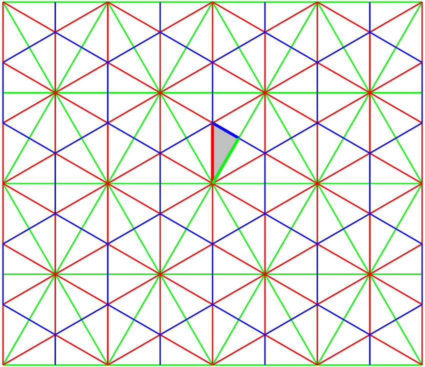

Examples:

- alcove $s_3 s_2 s_1 s_2 s_3 A_0$.

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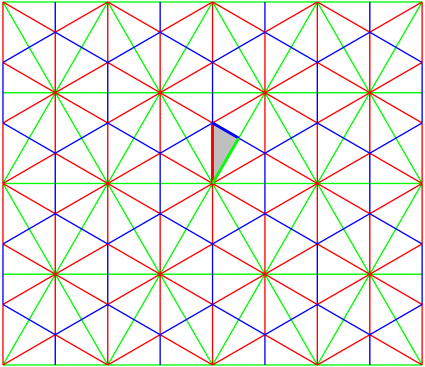

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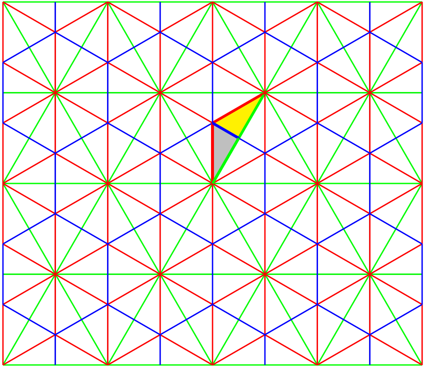

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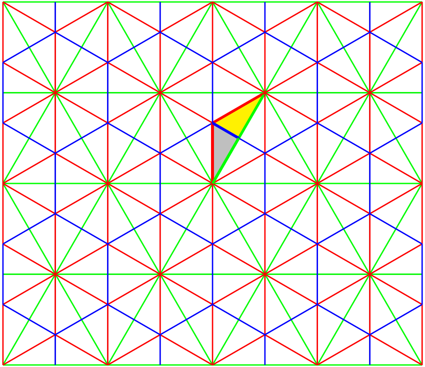

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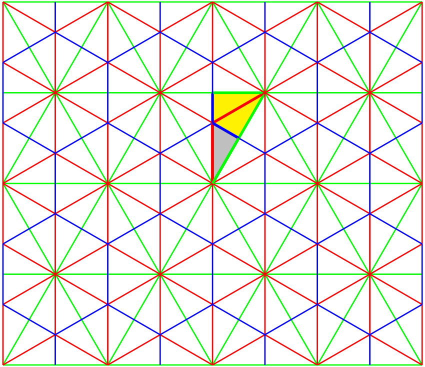

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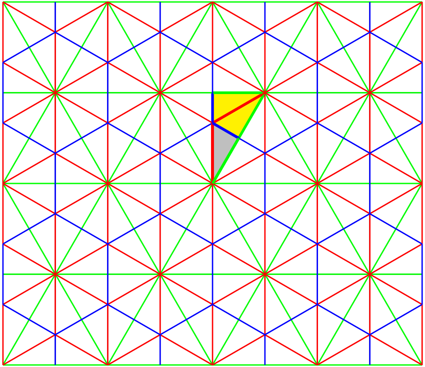

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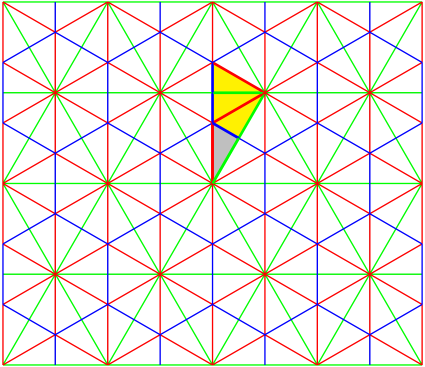

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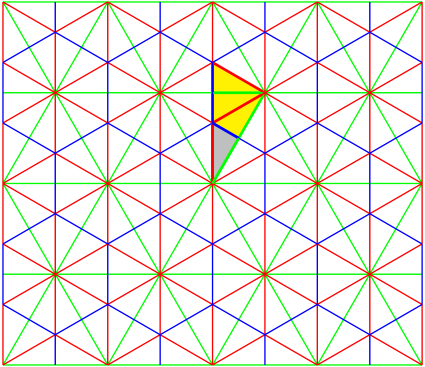

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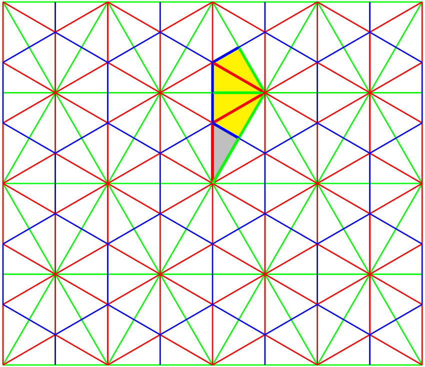

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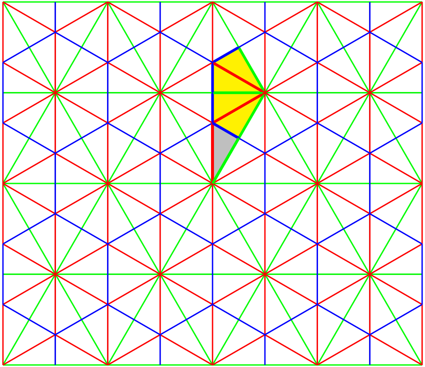

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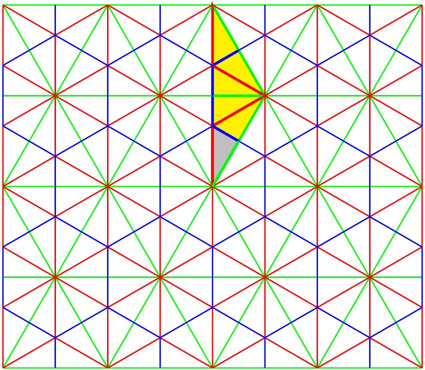

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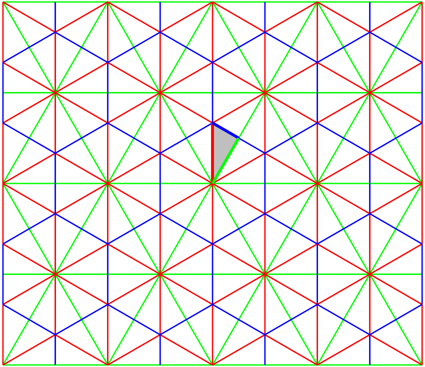

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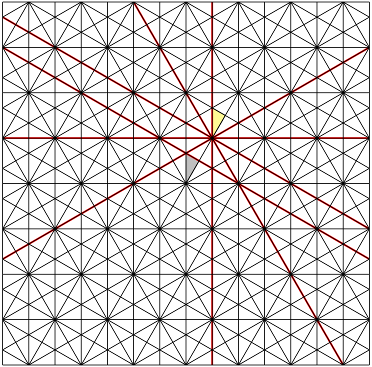
- We have $(s_1 s_3)^2 = e$.

Let $s, t \in S$. If a hyperplane H supports a face of type s and a face of type t then s and t are conjugate in W. Therefore we can associate to any hyperplane H a weight $c_H = L(s)$ if H supports a face of type s.

Let $w \in W$. $\ell(w)$ = number of hyperplanes which separates A_0 and wA_0

Let $u_1 \in W$ be a translation. Say $u_1 = s_2 s_1 s_2 s_1 s_2 s_3$.



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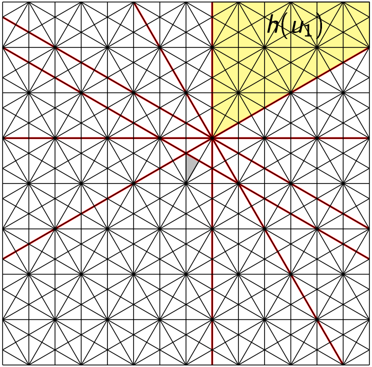
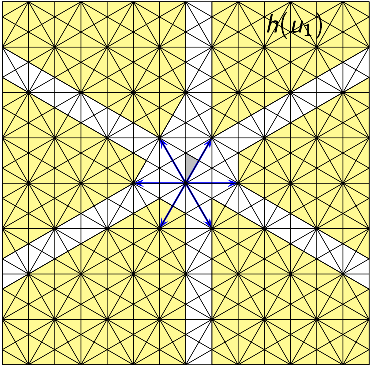


Let $z \in W$ such that $z.u_1$

Then, $z(u_1 A_0)$ has to be in $h(u_1)$.

Let $w \in W$. $\ell(w)$ = number of hyperplanes which separates A_0 and wA_0

Let $u_1 \in W$ be a translation. Say $u_1 = s_2 s_1 s_2 s_1 s_2 s_3$.



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Then, $z(u_1 A_0)$ has to be in $h(u_1)$.

Consider the orbit of $\vec{u_1}$ under Ω.

We look at the sets $h(u_i)$.

They are disjoint!

From there, one can see that

$z.u_i^r = z'.u_j^{r+m} \iff i = j$ and $z = z'.u_j^m$.

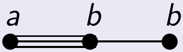
Theorem. G. ($\sim$ 2007)

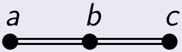
Let u be a translation. For $r \in \mathbf{N}$ large enough, we have

$$P_{z_1.u_i^r, z_2.u_j^r} = P_{z_1.u_i^{r+n}, z_2.u_j^{r+n}}$$

Theorem. G. ($\sim$ 2009)

Determination of the cells for W of type

$\tilde{G}_2$ 

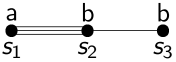
$\tilde{B}_2$ 

$(a, b, c > 0)$

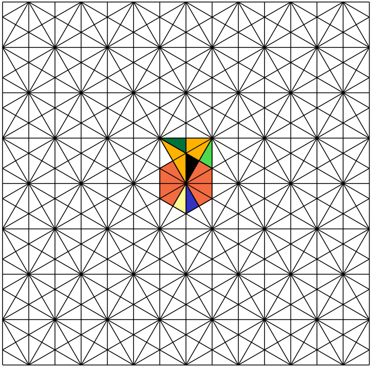
Furthermore, for all $x, y \in W$ we have

$$x \leq_{\mathcal{L}} y \text{ and } x \sim_{\mathcal{LR}} y \text{ then } x \sim_{\mathcal{L}} y$$

In fact, if W is an irreducible affine Weyl group, then there is an algorithm (based on Lusztig's conjectures) for determining the two-sided cells of W from the knowledge of the two-sided cells of all proper (hence, finite) parabolic subgroups.



$a > 3b$



cells in parabolic subgroups	**a**-function
$b_6 = \{e\}$	0
$b_5 = W_{2,3} - \{w_{2,3}, e\}$	b
$b_4 = \{w_{2,3}\}$	$3b$
$b_3 = W_{1,2} - \{e, s_2, s_1s_2s_1s_2s_1, w_{1,2}\}$	a
$b_2 = \{w_{1,3}\}$	$a + b$
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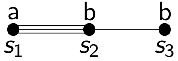

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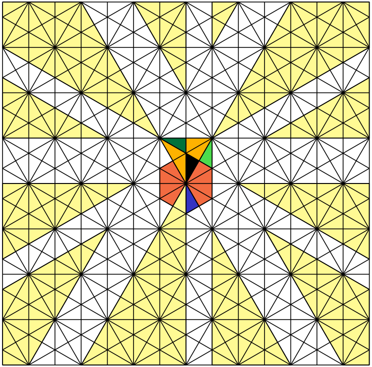

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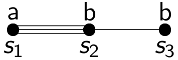

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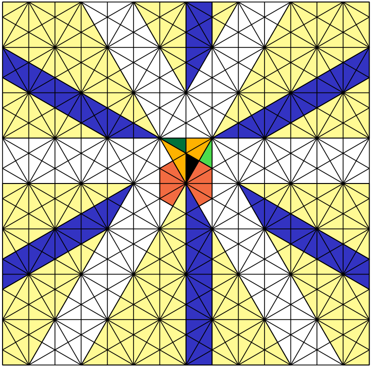

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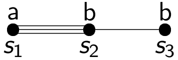

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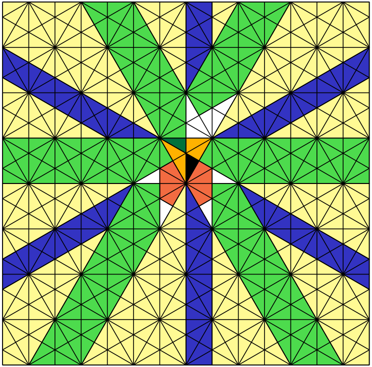

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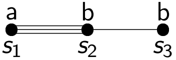

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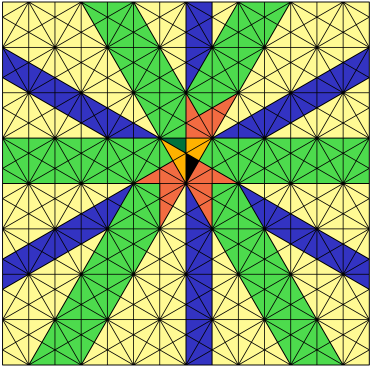

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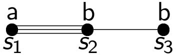
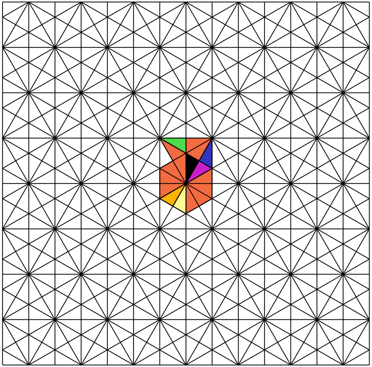


$a < b$



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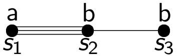

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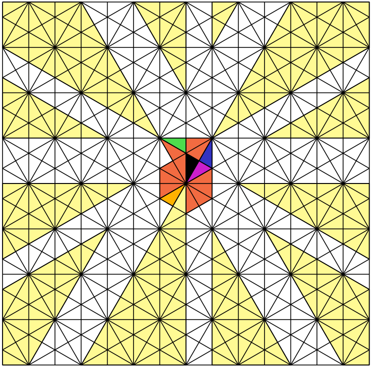

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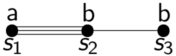

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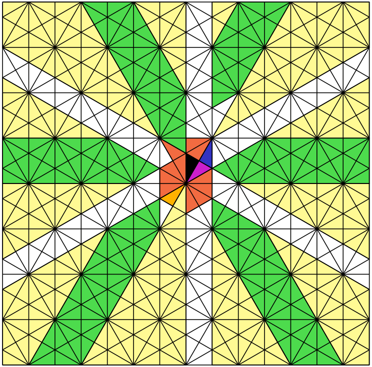

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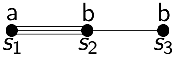

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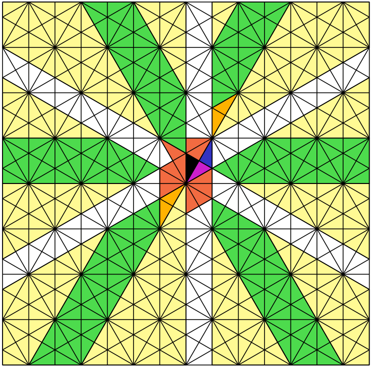

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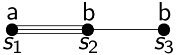

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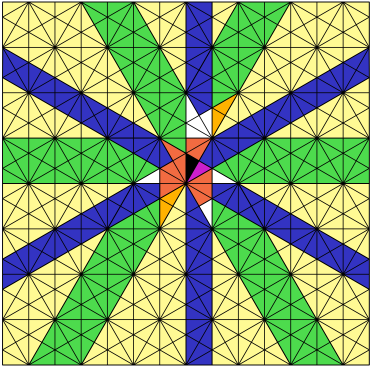

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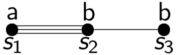

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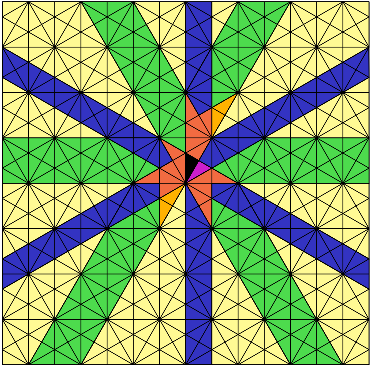

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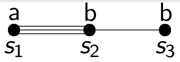
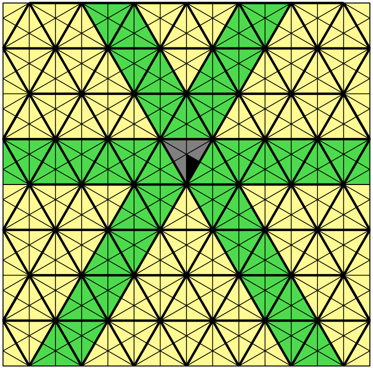


Let L such that $L(s_1) = a > 0$ and $L(s_2) = 0$

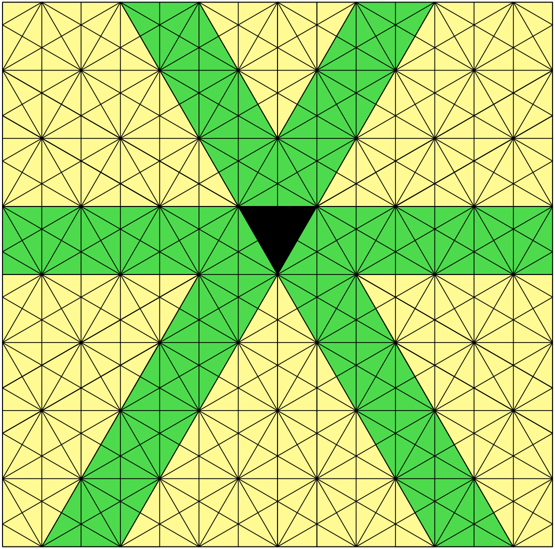


$W := W_{s_2,s_3} \ltimes \tilde{W}$

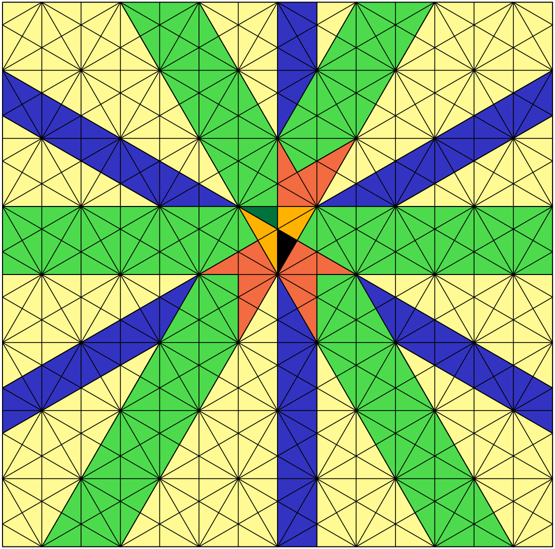
where $\tilde{W}$ is of type $\tilde{A}_2$ and generated by $\{s_1, s_2s_1s_2, s_3s_2s_1s_2s_3\}$.

The two-sided cells of W, L are...

$$a > 0, b = 0$$



$$a/b > 2$$



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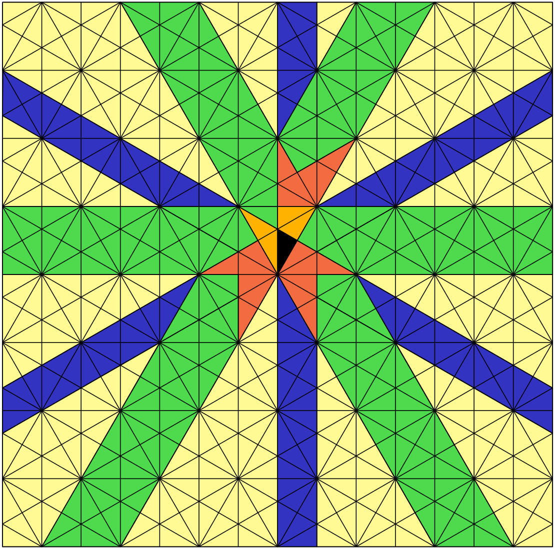
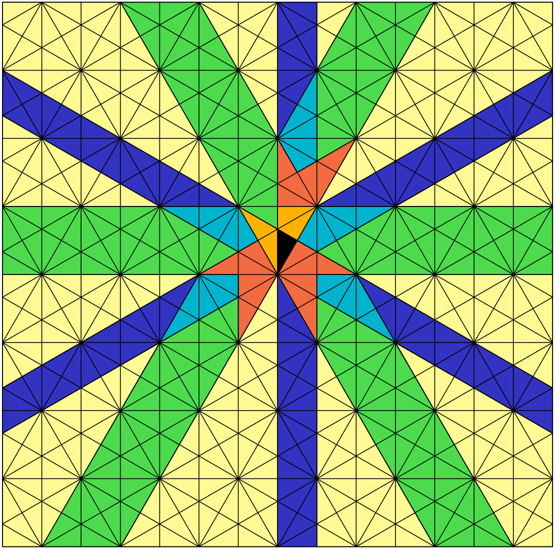
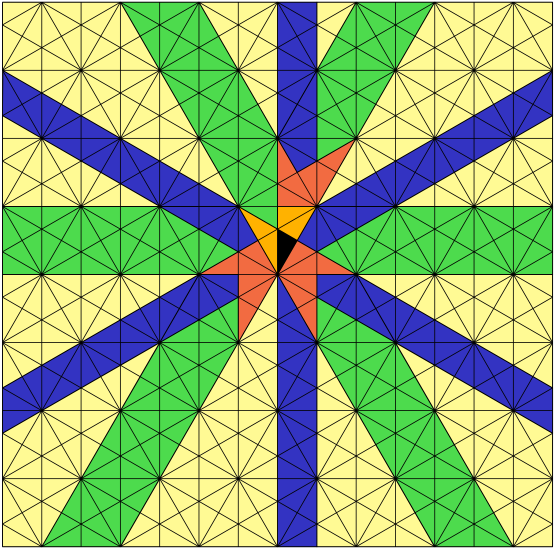


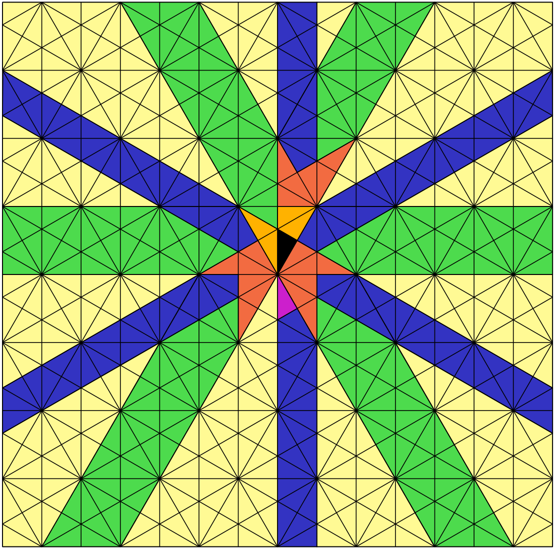
$$2 > a/b > 3/2$$



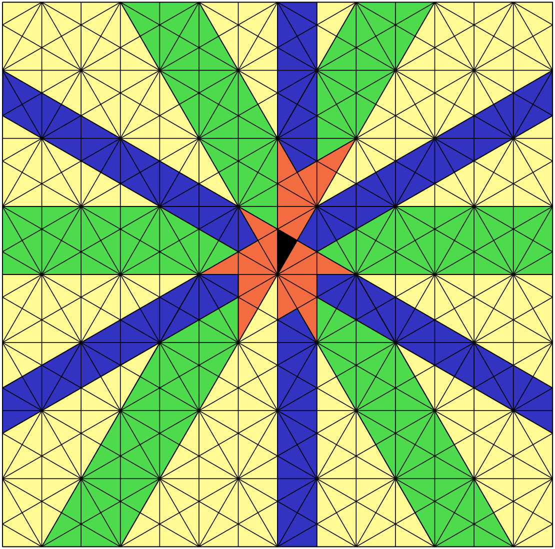
$$a/b = 3/2$$



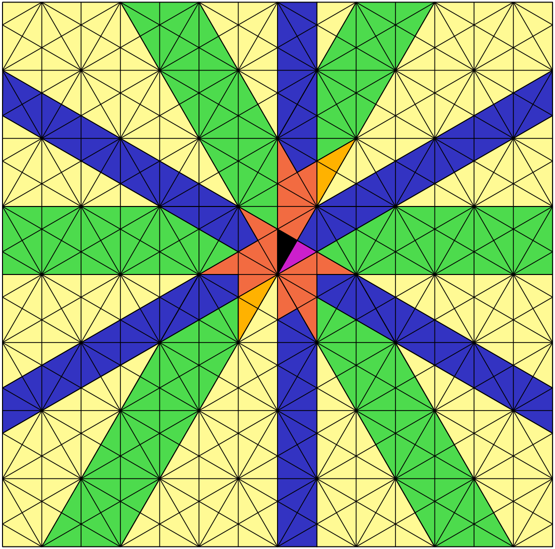
$3/2 > a/b > 1$



$$a/b = 1$$



$a/b < 1$



$a = 0, b > 0$